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Smoothing Effect and Propagations of Singularities for Viscoelastic Plates

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We study dissipative models for plates and we show that the solutions have a smoothing effect on the initial data for viscous plates, while for materials with memory the solution propagates singularities, that is, the solution of the plate equation of memory type is as regular as the initial data. Moreover, we show that when both dissipations are present, the memory type prevails in the sense that the solution propagates singularities. Finally, we prove the existence of global solutions for non-linear dissipative equations, with small data, which decay exponentially as time goes to infinity. © 1997 Academic Press

1. INTRODUCTION

In the motion of elastic bodies dissipation of the energy occurs by three means. First, when the temperature is different at different points in the body. This fact produces the irreversible process of thermal conduction, which makes the dissipation of the energy negative proportional to the square of the thermal difference. The corresponding equation for plates is known as the thermoelastic plate equation and is written in its simple form

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(isotropic and homogeneous plates) as

$$u_{tt} + \Delta^2 u + \alpha \Delta \theta = 0 \qquad \text{in } \Omega$$
$$\theta_t - \Delta \theta + \alpha \Delta u_t = 0 \qquad \text{in } \Omega,$$
$$u = \Delta u = \theta = 0, \qquad \text{on } \partial\Omega,$$

where u is the transversal displacement and θ is the difference of temperature. Second, when internal motion occurs in the body there are irreversible processes arising from the finite velocity of that motion. This means that the energy dissipation is produced by the internal friction or viscosity. In this situation the approximated linear plate model is given by

$$u_{tt} + \Delta^2 u + \alpha \Delta u_t = 0 \qquad \text{in } \Omega. \tag{1.1}$$

Finally, dissipation of the energy is produced also when the stresses at any instant depend on the history of strains which the material has undergone. In this case we say that the dissipation of energy is provided by the memory effect of the body. The equation of motion which reflects this property is written as

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\cdot, \tau)\, d\tau = 0 \qquad \text{in } \Omega. \tag{1.2}$$

The boundary condition we consider here is

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where g is the derivative of the relaxation function, which characterizes the memory of the material. Note that if $g = 0$ then we get the simple plate equation.

In this paper we compare the dissipation produced by the internal motion and the memory effect. That is, we study the behaviour of the solution of Eqs. (1.1) and (1.2) which these damping mechanisms produce. We show that the dissipation given by the internal friction is strong enough to produce a uniform rate of decay (exponentially in bounded domains) and a smoothing effect on the initial data, that is, the solution of the viscoelastic equation (1.1) is C^∞ at any positive time, no matter how irregular the initial data are. While if we consider the dissipation given by the memory effect (Eq. (1.2)) although we also can show an exponential rate of decay, the arbitrary smoothing effect on the initial data does not hold any more. This means that the solution of (1.2) propagates singularities in the sense that if the initial data do not belong to H^m, then the corresponding solution does not belong to H^{m+1}. So, there exists a regularizing effect on one order only. From our above discussion it seems that the dissipation given by internal friction is stronger than that produced by the memory effect. So we may ask: What happens with materials for which both kinds of dissipation are present? That is, we study the behaviour of the solution of the equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\cdot,\tau)\,d\tau - \gamma\Delta u_t = 0, \tag{1.3}$$
$$u = \Delta u = 0 \qquad \text{on } \partial\Omega.$$

Does the solution propagate singularities like the wave equation? Does the solution have the smoothing effect property like the heat equation? Or is it not possible to get any characterization? We will show in Sections 2 and 3 that, contrary to our intuition, this kind of equation propagates singularities, which means that the memory effect prevails over the internal friction, making the corresponding equation exhibit hyperbolic behaviour.

We will show the above properties in a general setting, where viscoelastic plate equations appear as a particular case. Energy estimates are also important to prove decay rates for solutions as time t tends to infinity. To be more precise, let us introduce the following notations:

- (H1) Let us denote by A a selfadjoint, positive operator such that $D(A^r) \subset D(A^s)$ with compact embedding, for $r > s \geq 0$, in a Hilbert space H equipped with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

- (H2) Let us denote by M, N, S smooth, real-valued C^2-functions, such that $M(0) > 0$, $N(0) > 0$, $S(0) \geq 0$, and $g \in C^3(\mathbb{R})$.

In Sections 2 and 3 we investigate the equation

$$u_{tt} + M([u])Au + S([u])A^{\beta}u_t - \int_0^t g(t-\tau)N([u])Au\,d\tau = 0 \tag{1.4}$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1.$$

By $[u]$ we are denoting the vector field

$$[u(t)] = \left(\|u_t\|^2, \|A^{1/2}u\|^2, (u,u_t)\right). \tag{1.5}$$

Concerning the existence of the solution, we show that Eq. (1.4) is well posed for small data in the norm of $D(A^{1/2}) \times H$. That is, if we take initial data (u_0, u_1) small in $D(A^{1/2}) \times H$, then we can prove the existence of the global solution which decays exponentially as time goes to infinity. Concerning the regularity of the solution, we show the smoothness effect on the initial data, which means that no matter how irregular the initial data are (with initial data such that the existence of at least a local solution is guaranteed) the solution must be in $C^\infty(0,T;D(A^\infty))$, provided $N=0$, $0<\beta<2$, M and S are C^∞-functions. By $D(A^\infty)$ we are denoting $\bigcap_{i=1}^\infty D(A^i)$. So, the corresponding equation has the same behaviour as the heat equation. While if we take $N\neq 0$, then we can show that the solution propagates singularities, which means that the solution is as regular as the initial data. So, the equation has an hyperbolic behaviour.

2. EXISTENCE AND ASYMPTOTIC BEHAVIOUR

In this section we prove the existence of global solutions of Eq. (1.4) for small data (u_0,u_1) in $D(A^{1/2})\times H$. Our proof is based on a priori estimates which we use to continue the local solution globally in time. The existence of a local smooth solution to (1.4) is established using the standard contraction mapping argument and we omit details here (cf. [12]).

THEOREM 2.1. *Let us take* $(u_0,u_1)\in D(A)\times D(A^\theta)$, *where* $\theta=\max\{1/2,\beta\}$. *Then there exists* $T>0$, *and a function* $u=u(t)$ *solution of Eq.* (1.4) *satisfying*

$$u\in C^2([0,T[;H)\cap C^1([0,T[;D(A^\theta))\cap C([0,T[;D(A)).$$

To prove the existence of the global solution we will suppose that $S=0$. For $S>0$ there exists a large literature; see, for example, Nishihara [11], Ikehata [7], and Matos and Pereira [9], among others. So the resulting equation is given by

$$u_{tt}+M([u])Au-\int_0^t g(t-\tau)N([u])Au(\tau)\,d\tau=0 \tag{2.1}$$

$$u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x).$$

We will use the following hypotheses on g

(G1) $g(t)>0$, $-kg(t)\le g'(t)\le -cg(t)$, $0\le g''(t)\le Cg(t)$, $-c_0g(t)\le g'''(t)\le 0$, $\forall t\ge 0$,

(G2) $M(0)-N(0)\int_0^\infty g(\tau)\,d\tau>0$,

where k,c,C,c_0 are positive constants. Let us take $g\in C(\mathbb{R})$, $f\in C^1([0,T];H)$, and $\eta\in C(\mathbb{R})$. We introduce the notation

$$(g\,\square\, f)(t)=\int_0^t g(t-\tau)\|f(\tau)-f(t)\|^2\,d\tau$$

and

$$(\eta * v)(t)=\int_0^t \eta(t-\tau)v(\tau)\,d\tau.$$

It is not difficult to show that

$$2(\eta * \phi, \phi') = -\eta(t)\|\phi\|^2 - \frac{d}{dt}\left\{\eta \,\square\, \phi - \left(\int_0^t \eta \, d\tau\right)\|\phi\|^2\right\} + \eta' \,\square\, \phi, \tag{2.2}$$

provided η is a C^1-function and $\phi \in C^1([0,T]; H)$. To explore the dissipative properties of the memory effect we rewrite Eq. (2.1) as

$$u_{tt} + M(0)Au - N(0)\int_0^t g(t-\tau)Au(\tau)\, d\tau = P \stackrel{\text{def}}{=} Q + R,$$

where

$$R(t) = \int_0^t g(t-\tau)\{N([u](\tau)) - N(0)\}Au(\tau)\, d\tau \tag{2.3}$$

$$Q(t) = \{M(0) - M([u(t)])\}Au.$$

From the hypotheses on A it follows that there exists a sequence of eigenfunctions $(w_i)_{i \in \mathbb{N}}$ and eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$ of A such that the eigenfunctions are a basis of H while the eigenvalues satisfy

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \lambda_i \to \infty \qquad \text{as } i \to \infty.$$

Writing

$$u(t) = \sum_{j=1}^{\infty} h_j(t) w_j$$

we get

$$\|A^s u(t)\|^2 = \|u(t)\|^2_{D(A^s)} := \sum_{j=1}^{\infty} \lambda_j^{2s} h_j^2(t), \qquad \forall s \in \mathbb{R}.$$

When $s < 0$, by $D(A^s)$ we denote $D(A^s) = [D(A^{-s})]'$ the dual space of $D(A^{-s})$. From now on we will suppose, without loss of generality, that $M(0) = 1$ and $N(0) = 1$ (to see this we make the change $t \mapsto \sqrt{M(0)}\,t$ then consider $\hat{g} := (N([0])/M([0]))g$) then hypotheses (G2) can be written as

$$1 - \int_0^{\infty} g(\tau)\, d\tau = \alpha > 0. \tag{2.4}$$

The main difficulty in proving the global existence for small data in $D(A^{1/2}) \times H$ is to show that the first order energy decays exponentially; but we overcome it by using the spectral properties of the operator A.

Let us consider the spectral equation associated to (2.3), obtained by applying the inner product with w_j to Eq. (2.3); that is,

$$h_j'' + \lambda_j h_j - \lambda_j \int_0^t g(t-\tau) h_j(\tau)\, d\tau = p_j := q_j + r_j, \tag{2.5}$$

$$h_j(0) = (u_0, w_j); \qquad h_j'(0) = (u_1, w_j),$$

where

$$q_j = \{M(0) - M([u])\} \lambda_j h_j$$

$$r_j = \lambda_j \int_0^t g(t-\tau) \{N([u]) - N(0)\} h_j(\tau)\, d\tau.$$

Let us introduce the functions

$$\varepsilon_j(t) = \frac{1}{2}\left\{ |h_j'|^2 + \lambda_j \left(1 - \int_0^t g\, d\tau\right) |h|^2 + \lambda_j g \,\Box\, h_j \right\}$$

$$K_j^1(t) = \frac{1}{2}|h_j''|^2 + \frac{\lambda_j}{2}|h_j'|^2 - \lambda_j \int_0^t f(t-\tau) h_j(\tau)\, d\tau\, h_j' + \frac{\lambda_j}{2} f(0) |h_j|^2$$

$$- \frac{\lambda_j}{2} f' \,\Box\, h_j + \frac{\lambda_j}{2}\left(\int_0^t f'\, d\tau\right)|h_j|^2$$

$$K_j^2(t) = h_j'' h_j' + \frac{g(0)}{2}|h_j'|^2 + \frac{\lambda_j}{2} f \,\Box\, h_j - \frac{\lambda_j}{2}\left(\int_0^t f\, d\tau\right)|h_j|^2$$

$$K_j^3(t) = h_j' h_j,$$

where $f = g(0)g + g'$. With these notations we have the following lemma

LEMMA 2.1. *Let us suppose that* $g \in C^3$ *and satisfies* (G1). *Then we have*

$$\frac{d}{dt}\left\{ K_j^1(t) + \frac{g(0)}{2} K_j^2(t) + \frac{g(0)}{4} \lambda_j K_j^3(t) \right\}$$

$$\leq -\frac{g(0)\alpha}{8}\left\{ |h_j''|^2 + \lambda_j |h_j'|^2 + \lambda_j^2 |h_j|^2 \right\} + \left[\frac{C}{2} + \frac{g(0)k}{2}\right] \lambda_j g |h_j|^2$$

$$+ \left(\frac{c_0}{2} + \frac{g(0)C}{4}\right) \lambda_j g \,\Box\, h_j + \frac{g(0)}{8\alpha} \lambda_j^2 g \,\Box\, h_j + \mathscr{R}_j(t),$$

where

$$\mathscr{R}_j^1(t) = \left\{ p_j' h_j'' + \frac{g(0)}{2} p_j' h_j' + \frac{g(0)}{4} \lambda_j p_j h_j + g(0) p_j h_j'' + \frac{g(0)^2}{2} p_j h_j' \right\}.$$

Proof. Note that if the right hand side of (2.5) is a C^1-function, then the function h_j is a C^3-function. Differentiating in time Eq. (2.5) and substituting the term $\lambda_j h_j$ given by Eq. (2.5) yield

$$h_j''' + \lambda_j h_j' + g(0) h_j'' - \lambda_j \int_0^t f(t-\tau) h_j(\tau)\, d\tau = p_j' + g(0) p_j. \quad (2.6)$$

Multiplying (2.6) by h_j'' and using

$$\lambda_j \int_0^t f(t-\tau) h_j(\tau)\, d\tau\, h_j'' = \lambda_j \frac{d}{dt}\left\{ \int_0^t f(t-\tau) h_j(\tau)\, d\tau\, h' \right\}$$
$$- \frac{\lambda_j f(0)}{2} \frac{d}{dt} |h_j|^2 - \lambda_j \int_0^t f'(t-\tau) h_j(\tau)\, d\tau\, h_j'$$

we get

$$\frac{1}{2}\frac{d}{dt}\left\{ |h_j''|^2 + \lambda_j |h_j'|^2 - 2\lambda_j \int_0^t f(t-\tau) h_j(\tau)\, d\tau\, h' + \lambda_j f(0) |h_j|^2 \right\}$$
$$= -g(0)|h_j''|^2 + p_j' h_j'' + g(0) p_j h_j'' - \lambda_j \int_0^t f'(t-\tau) h_j(\tau)\, d\tau\, h_j'.$$

From (2.2) and the above identity we get

$$\frac{d}{dt} K_j^1(t) = -g(0)|h_j''|^2 + \frac{\lambda_j f'}{2} |h_j|^2 - \frac{\lambda_j}{2} f'' \Box\, h_j + p_j' h_j'' + g(0) p_j h_j''.$$

Using (G1), the above identity can be written as

$$\frac{d}{dt} K_j^1(t) \le -g(0)|h_j''|^2 + \frac{\lambda_j}{2} Cg |h_j|^2 + \lambda_j c_0 g \Box\, h_j + p_j' h_j'' + g(0) p_j h_j''. \quad (2.7)$$

Multiplying (2.6) by h_j' we get

$$\frac{d}{dt} h_j'' h_j' = h_j''' h_j' + |h_j''|^2$$
$$= -\lambda_j |h_j'|^2 - g(0) h_j'' h_j' + \lambda_j \int_0^t f(t-\tau) h_j(\tau)\, d\tau\, h_j'$$
$$+ |h_j''|^2 + p_j' h_j' + g(0) p_j h_j',$$

and from (2.2) and hypothesis (G1) we get

$$\frac{d}{dt} K_j^2(t) \le |h_j''|^2 - \lambda_j |h_j'|^2 + \frac{\lambda_j}{2} Cg \Box\, h_j + k \lambda_j g |h_j|^2 + p_j' h_j' + g(0) p_j h_j'. \quad (2.8)$$

On the other hand, multiplying (2.5) by h_j we have

$$h_j''h_j = -\lambda_j|h_j|^2 - \lambda_j\int_0^t g(t-\tau)h_j(\tau)\,d\tau\,h_j + p_jh_j. \tag{2.9}$$

So recalling the definitions of $K_j^3(t)$ and using (2.9) we get

$$\frac{d}{dt}K_j^3(t) \le |h_j'|^2 - \lambda_j\left(1 - \int_0^t g\,d\tau\right)|h_j|^2$$
$$+ \lambda_j\int_0^t g(t-\tau)\big[h_j(\tau) - h_j(t)\big]\,d\tau\,h_j + p_jh_j.$$

Since $\alpha < 1 - \int_0^t g\,d\tau$ and $\int_0^\infty g\,d\tau < 1$ we have

$$\frac{d}{dt}K_j^3(t) \le |h_j'|^2 - \frac{\lambda_j\alpha}{2}|h_j|^2 + \frac{\lambda_j}{2\alpha}g\,\Box\,h_j + p_jh_j. \tag{2.10}$$

Finally, from (2.7), (2.8), and (2.10) our conclusion follows. ∎

LEMMA 2.2. *Under the same hypotheses as Lemma* 2.1 *and for any* $\delta \in]0,1[$ *such that*

$$\big|M([0]) - M([u](t))\big| < \delta \qquad \forall t \in [0, t_*]$$
$$\big|N([u](t)) - N([0])\big| < \delta \qquad \forall t \in [0, t_*],$$

we have

$$\lambda_j|p_jh_j| \le 3\delta\lambda_j^2h_j^2 + \frac{\delta}{2}\lambda_j^2 g\,\Box\,h_j, \tag{2.11}$$

$$p_jh_j' \le -\frac{1}{2}\left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_jh_j^2 + \delta\left(\frac{3k}{2} + g(0)\right)$$
$$\times\left(\lambda_j g\,\Box\,h_j + \lambda_j|h_j|^2\right) + \frac{d}{dt}\left\{\frac{1}{2}q_jh_j + r_jh_j\right\}, \tag{2.12}$$

$$p_j'h_j'' \le -\frac{1}{2}\left[\frac{d}{dt}\{M([0]) - N([u])\}\right]\lambda_j^2|h_j|^2 + \delta\frac{3k}{2}\lambda_j^2 g\,\Box\,h_j$$
$$+ \delta\left(k + \frac{g(0)}{2}\right)|h_j''|^2 + \delta\left\{\frac{7k + 5g(0)}{2}\right\}\lambda_j^2|h_j|^2$$
$$+ \frac{d}{dt}\left\{-\frac{\lambda_j}{2}q_jh_j + \lambda_j(g * h_j)q_j + \frac{|q_j|^2}{2} + r_jq_j\right\}, \tag{2.13}$$

$$|p_jh_j''| \le \delta\lambda_j^2|h_j|^2 + \frac{3}{2}\delta|h_j''|^2 + \frac{\delta}{2}\lambda_j^2 g\,\Box\,h_j, \tag{2.14}$$

for any $t \in [0, t]$.

Proof. Recalling the definition of p_j we get

$$\begin{aligned}
\lambda_j p_j h_j &= \{M([0]) - M([u])\}\lambda_j^2 |h_j|^2 \\
&\quad + \lambda_j^2 \int_0^t g(t-\tau)\{N([u]) - N([0])\} h_j(\tau)\, d\tau\, h_j \\
&= \{M([0]) - M([u])\}\lambda_j^2 |h_j|^2 \\
&\quad + \lambda_j^2 \int_0^t g(t-\tau)\{N([u]) - N([0])\}\big[h_j(\tau) - h_j(t)\big]\, d\tau\, h_j \\
&\quad + \lambda_j^2 \int_0^t g(t-\tau)\{N([u]) - N([0])\}\, d\tau\, h_j^2.
\end{aligned}$$

From the hypotheses on the initial data we have

$$\begin{aligned}
|\lambda_j p_j h_j| &\le \delta\lambda_j^2 h_j^2 + \delta\lambda_j^2 \left(\int_0^t g\, d\tau\right)^{1/2} (g \,\square\, h_j)^{1/2} |h_j| + \lambda_j^2 \delta h_j^2 \left(\int_0^t g\, d\tau\right) \\
&\le 2\delta\lambda_j^2 h_j^2 + \frac{\delta\lambda_j^2}{2} g \,\square\, h_j + \frac{\delta\lambda_j^2}{2} |h_j|^2 \\
&\le 3\delta\lambda_j^2 h_j^2 + \frac{\delta\lambda_j^2}{2} g \,\square\, h_j
\end{aligned}$$

which proves (2.11). To show (2.12) note that $q_j h_j' = (d/dt)[q_j h_j] - q_j' h_j$ and

$$\begin{aligned}
q_j' h_j &= \left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j |h_j|^2 + [M([0]) - M([u])]\frac{\lambda_j}{2}\frac{d}{dt}|h_j|^2 \\
&= \left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j |h_j|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\Big[\{M([0]) - M([u])\}\lambda_j |h_j|^2\Big] \\
&\quad - \frac{1}{2}\left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j |h_j|^2 \\
&= \frac{1}{2}\left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j h_j^2 + \frac{1}{2}\frac{d}{dt}(q_j h_j). \qquad (2.15)
\end{aligned}$$

Then

$$q_j h_j' = \frac{1}{2}\frac{d}{dt}(q_j h_j) - \frac{1}{2}\left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j |h_j|^2. \quad (2.16)$$

On the other hand,

$$\begin{aligned} r_j h_j' &= \lambda_j \int_0^t g(t-\tau)\{N([u]) - N([0])\} h(\tau)\, d\tau\, h' \\ &= \lambda_j \frac{d}{dt}\left[\int_0^t g(t-\tau)\{N([u]) - N([0])\} h(\tau)\, d\tau\, h\right] \\ &\quad - \lambda_j \int_0^t g'(t-\tau)\{N([u]) - N([0])\} h_j(\tau)\, d\tau\, h_j \\ &\quad - \lambda_j g(0)\{N([u]) - N([0])\} h_j^2 \\ &= \frac{d}{dt}(r_j h_j) - \lambda_j \int_0^t g'(t-\tau)\{N([u]) - N([0])\}\left[h_j(\tau) - h_j(t)\right] d\tau\, h_j \\ &\quad - \lambda_j \int_0^t g'(t-\tau)\{N([u]) - N([0])\}\, d\tau\, h_j^2 - \lambda_j g(0)\{N([u])\} h_j^2. \end{aligned}$$

From the hypotheses of Lemma 2.2 and (G1) we get

$$r_j h_j' \le \frac{d}{dt}(r_j h_j) + \frac{\delta \lambda_j k}{2} g \,\Box\, h_j + \delta\left(\frac{3k}{2} + g(0)\right)\lambda_j |h_j|^2. \quad (2.17)$$

Using (2.16), (2.17), and recalling the definition of p_j we have

$$\begin{aligned} p_j h_j' \le{}& \frac{1}{2}\frac{d}{dt}(q_j h_j) + \frac{d}{dt}(r_j h_j) - \frac{1}{2}\left[\frac{d}{dt}\{M([0]) - M([u])\}\right]\lambda_j |h_j|^2 \\ &+ \delta\left(\frac{3k}{2} + g(0)\right)\left(\lambda_j g \,\Box\, h_j + \lambda_j |h_j|^2\right) \end{aligned}$$

which proves (2.12). To show inequality (2.13), let us use Eq. (2.5) to get

$$q_j' h_j'' = -\lambda_j h_j q_j' + \lambda_j \int_0^t g(t-\tau) h_j(\tau)\, d\tau q_j' + p_j q_j'.$$

From (2.16) and the relations

$$\lambda_j\int_0^t g(t-\tau)h_j(\tau)\,d\tau q_j' = \frac{d}{dt}\left\{\lambda_j\int_0^t g(t-\tau)h_j(\tau)\,d\tau q_j\right\} - \frac{d}{dt}\left\{\lambda_j\int_0^t g(t-\tau)h_j(\tau)\,d\tau\right\}q_j,$$

$$\begin{aligned}
&\frac{d}{dt}\left\{\lambda_j\int_0^t g(t-\tau)h_j(\tau)\,d\tau\right\}q_j\\
&\quad = \lambda_j^2\int_0^t g'(t-\tau)\{h_j(\tau)-h_j(t)\}\,d\tau\{M([0])-M([u])\}h_j(t)\\
&\qquad + \lambda_j^2\int_0^t g'(\tau)\,d\tau\, h_j^2(t)[M([0])-M([u])]\\
&\qquad + \lambda_j^2 g(0)[M([0])-M([u])]h_j^2\\
&\quad \le \frac{\delta k\lambda_j^2}{2}g\,\Box\,h_j + \delta\left(\frac{k}{2}+k+g(0)\right)\lambda_j^2|h_j|^2,
\end{aligned}$$

we conclude that

$$\begin{aligned}
q_j'h_j'' \le& -\frac{\lambda_j}{2}\frac{d}{dt}(q_jh_j) + \frac{\lambda_j^2}{2}\left[\frac{d}{dt}\{M([0])-M([u])\}\right]|h_j|^2\\
&+\frac{d}{dt}[(g*h_j)q_j] + \frac{\delta k\lambda_j^2}{2}g\,\Box\,h_j + 3k\delta\lambda_j^2|h_j|^2\\
&+2\delta g(0)\lambda_j^2|h_j|^2 + \frac{1}{2}\frac{d}{dt}|q_j|^2 + \frac{d}{dt}(r_jq_j) - r_j'q_j, \qquad (2.18)
\end{aligned}$$

$$\begin{aligned}
r_j'q_j =& \lambda_j\int_0^t g'(t-\tau)\{N([0])-N([u])\}\{h(\tau)-h(t)\}\,d\tau\\
&\times[M([0])-M([u])]\lambda_jh_j\\
&+\lambda_j\int_0^t g'(t-\tau)\{N([u])-N([0])\}\,d\tau[M([0])-M([u])]\lambda_jh_j^2\\
&+\lambda_jg(0)\{N([u])-N([0])\}h_j\{M([0])-M([u])\}\lambda_jh_j\\
\le& \frac{\lambda_j^2\delta^2k}{2}g\,\Box\,h_j + \frac{3\lambda_j^2\delta^2k}{2}|h_j|^2 + \lambda_j^2h_j^2g(0)\delta^2, \qquad (2.19)
\end{aligned}$$

and also that

$$
\begin{aligned}
r_j'h_j'' &= \lambda_j \int_0^t g'(t-\tau)h_j(\tau)\{N([u])-N([0])\}\,d\tau\, h_j'' \\
&\qquad + \lambda_j g(0)h_j h_j''\{N([u])-N([0])\} \\
&\le \lambda_j \delta k\left(\int_0^t g\,d\tau\right)^{1/2}(g\,\Box\,h_j)^{1/2}|h_j''|^2 + \lambda_j\delta k\left(\int_0^t g\,d\tau\right)|h_j|\,|h_j''| \\
&\qquad + \lambda_j g(0)\,\delta|h_j|\,|h_j''| \\
&\le \frac{\lambda_j^2}{2}\delta k g\,\Box\,h_j + \delta\left(k+\frac{g(0)}{2}\right)|h_j''|^2 + \delta\left(\frac{k}{2}+\frac{g(0)}{2}\right)\lambda_j^2|h_j|^2. \qquad (2.20)
\end{aligned}
$$

Recalling that $p_j' = q_j' + r_j'$, from (2.18), (2,19), and (2.20), we get inequality (2.13). Finally, let us prove (2.14). From the definition of p_j we have

$$
\begin{aligned}
p_j h_j'' &= \{M([0])-M([u])\}\lambda_j h_j h_j'' \\
&\qquad + \lambda_j\int_0^t g(t-\tau)\{N([u])-N([0])\}h_j(\tau)\,d\tau\,h_j'' \\
&\le \delta\lambda_j|h_j|\,|j''| + \delta\lambda_j\left(\int_0^t g\,d\tau\right)^{1/2}(g\,\Box\,h_j)^{1/2}|h_j''| \\
&\qquad + \delta\lambda_j\left(\int_0^t g\,d\tau\right)|h_j|\,|h_j''| \\
&\le \delta\lambda_j^2|h_j|^2 + \frac{3\delta}{2}|h_j''|^2 + \frac{\delta}{2}\lambda_j^2 g\,\Box\,h_j,
\end{aligned}
$$

and our conclusion follows. The proof is now complete. ∎

LEMMA 2.3. *Under the same hypotheses as Lemma* 2.2, *we have*

$$|h_j''|^2 \le 17\left(\lambda_j^2 h_j^2 + \lambda_j^2 g\,\Box\,h_j\right), \qquad (2.21)$$

for $0 < \delta < 1/6$.

Proof. Multiplying Eq. (2.5) by h_j'' we have

$$
\begin{aligned}
|h_j''|^2 &= -\lambda_j h_j h_j'' + \lambda_j\int_0^t g(t-\tau)\left[h_j(\tau)-h_j(t)\right]d\tau\,h_j'' \\
&\qquad + \lambda_j\left(\int_0^t g\,d\tau\right)h_j''h_j + p_j h_j'' \\
&\le 2\lambda_j|h_j|\,|h_j''| + \lambda_j\left(g\,\Box\,h_j\right)^{1/2}|h_j''| + |p_j h_j''|.
\end{aligned}
$$

From Lemma 2.2 we get

$$\begin{aligned} |h_j''|^2 &\leq (4+\delta)\lambda_j^2|h_j|^2 + \left(1+\frac{\delta}{2}\right)\lambda_j^2 g\,\Box\,h_j + \left(\frac{1}{2}+\frac{3\delta}{2}\right)|h_j''|^2 \\ &\leq \frac{25}{6}\left[\lambda_j^2|h_j|^2 + \lambda_j^2 g\,\Box\,h_j\right] + \frac{3}{4}|h_j''|^2. \end{aligned}$$

Therefore

$$|h_j''|^2 \leq 17\left[\lambda_j^2|h_j|^2 + \lambda_j^2 g\,\Box\,h_j\right].$$

Then our conclusion follows. ▮

To get the global existence of solutions for Eq. (2.1), we define the functionals

$$\begin{aligned} \mathscr{L}_j(t) &= K_j^1(t) + \frac{g(0)}{2}K_j^2(t) + \frac{g(0)}{4}\lambda_j K_j^3(t) + N_1(1+\lambda_j)\varepsilon_j(t) \\ \mathscr{M}_j(t) &= |h_j''|^2 + \lambda_j|h_j'|^2 + \lambda_j^2 g\,\Box\,h_j + \lambda_j^2|h_j|^2 \\ \mathscr{S}_j(t) &= \frac{\lambda_j}{2}q_jh_j + \lambda_j(g*h_j)q_j + \frac{|q_j|^2}{2} + r_jq_j + \frac{g(0)}{2}p_jh_j' \\ &\quad + \left[N_1(1+\lambda_j) + \frac{g(0)^2}{2}\right]\left[\frac{q_jh_j}{2} + r_jh_j\right] \\ E_1(t) &= \|u_t\|^2 + \|A^{1/2}u\|^2 + g\,\Box\,A^{1/2}u \\ E_2(t) &= \|u_{tt}\|^2 + \|A^{1/2}u_t\|^2 + \|Au\|^2, \end{aligned}$$

where N_1 is chosen such that there exist positive constants c_0 and c_1 satisfying

$$c_0\mathscr{M}_j(t) \leq \mathscr{L}_j(t) \leq c_1\mathscr{M}_j(t). \tag{2.22}$$

Under these conditions we have

THEOREM 2.2. *Under the same hypotheses as Lemma* 2.1, *there exists* $\varepsilon > 0$ *such that for any*

$$(u_0, u_1) \in D(A) \times D(A^{1/2})$$

satisfying

$$\|A^{1/2}u_0\|^2 + \|u_1\|^2 < \varepsilon,$$

there exists only one solution u of Eq. (2.1) *such that*

$$u \in C^2([0,\infty[;H) \cap C^1([0,\infty[;D(A^{1/2})) \cap C([0,\infty[;D(A)),$$

which decays as

$$E_1(t) \leq E_1(0)e^{-\gamma t} \qquad \forall t \geq 0 \text{ and } \gamma > 0.$$

Proof. Multiplying Eq. (2.5) by h_j' and using identity (2.2) we have

$$\frac{d}{dt}\varepsilon_j(t) \leq -\frac{\lambda_j}{2}g|h_j|^2 - \frac{c\lambda_j}{2}g \,\Box\, h_j + p_j h_j'. \tag{2.23}$$

From Lemma 2.1 and relation (2.23) and taking

$$N_1 \geq \max\left\{C + kg(0) + \frac{g(0)\alpha}{4}, g(0)\left[C + \frac{\alpha}{2}\right], g(0)\left[\frac{1}{4\alpha c} + \frac{\alpha}{8c}\right]\right\}$$

we get

$$\frac{d}{dt}\mathscr{L}_j(t) \leq \frac{g(0)\alpha}{8}\mathscr{M}_j(t) + \mathscr{R}_j(t), \tag{2.24}$$

where

$$\mathscr{R}_j(t) = \mathscr{R}_j^1(t) + N_1(1 + \lambda_j)p_j h_j'.$$

Since M and N are continuous functions, we have that for any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|\sigma|_{\mathbb{R}^3} < c_2\varepsilon \Rightarrow |M(\sigma) - M([0])| < \delta \qquad \text{and} \qquad |N(\sigma) - N([0])| < \delta,$$

where $c_1 > \max\{d, d\theta_0\}$, θ_0 is the embedding constant of $D(A^{1/2})$ in H, and $d = 72c_1/c_0$. Taking

$$E_1(0) = \|A^{1/2}u_0\|^2 + \|u_1\|^2 < \varepsilon,$$

$$E_2(0) = \|u_{tt}(0)\|^2 + \|A^{1/2}u_t(0)\|^2 + \|Au_0\|^2 < \mu, \qquad \mu = \mu(u_0, u_1) > 1,$$

from Theorem 2.1 there exists $0 < T_0 \leq T_{\max}$, such that

$$E_1(t) \leq d\varepsilon \qquad \text{and} \qquad E_2(t) \leq d\mu \text{ in } [0, T_0[.$$

Let us denote

$$T_1^* = \sup\{t_1 > 0: E_1(t) < d\varepsilon \text{ in } [0, t_1[\}$$

$$T_2^* = \sup\{t_2 > 0: E_2(t) < d\mu \text{ in } [0, t_2[\}$$

and

$$t^* = \min\{T_1^*, T_2^*\}.$$

We have two possibilities: (i) $T^* = T_{\max}$, (ii) $T^* < T_{\max}$. The first one implies that the solution u is bounded so we have $T_{\max} = \infty$. So, we only consider case (ii). Let us suppose that $T^* < T_{\max}$ and $T_{\max} < \infty$. Since $E_1(t) < d\varepsilon$ in $[0, T^*[$ it follows that

$$\|u_t(t)\|^2 < d\varepsilon < c_2\varepsilon, \qquad \|A^{1/2}u\|^2 < d\varepsilon < c_2\varepsilon$$

and

$$|(u, u_t)| \leq \|u(t)\| \, \|u_t(t)\| \leq \theta_0 \|A^{1/2}u\| \, \|u_t(t)\| \leq \theta_0 \, d\varepsilon < c_2\varepsilon,$$

and from this we get

$$|[u(t)]|_{\mathbb{R}^3} < c_2\varepsilon \Rightarrow |M([u]) - M([0])| < \delta$$

and

$$|N([u]) - N([0])| < \delta \text{ in } [0, T^*[. \tag{2.25}$$

Let us denote

$$\nu = \max_{|\sigma| \leq c_2\varepsilon} \left\{ \frac{\partial M}{\partial x_i}(\sigma) : i = 1, 2, 3 \right\}.$$

Then we have

$$\begin{aligned} &\left| \frac{d}{dt}\{M([0]) - M([u])\} \right| \\ &\qquad \leq 2\nu\left\{ |(u_t(t), u_{tt}(t))| + \left|\left(A^{1/2}u, A^{1/2}u_t\right)\right| + \|u_t(t)\|^2 + |(u, u_{tt})| \right\} \\ &\qquad \leq 2\nu \|u_t(t)\| \{\|u_{tt}(t)\| + \|Au(t)\| + \|u_t(t)\| + \|u(t)\|\} \\ &\qquad \leq c_3\sqrt{\varepsilon}, \end{aligned} \tag{2.26}$$

where $c_3 = 8\nu\sqrt{\mu}\,c_2 > 0$. From (2.25), (2.26), and Lemma 2.2 we get

$$\mathscr{R}(t) \leq \delta c_4 \mathscr{M}_j(t) + \left(\frac{1}{2} + \frac{g(0)^2}{4} + N \right) c_3\sqrt{\varepsilon}\left(\lambda_j^2 h_j^2\right) + \frac{d}{dt}\mathscr{S}_j(t)$$

$$|\mathscr{S}_j(t)| \leq \delta c_5 \mathscr{M}_j(t)$$

for c_4 and c_5 positive constants. Taking ε and δ small enough ($\delta < 1/6$) it follows from (2.24) that

$$\frac{d}{dt}\left\{\mathscr{L}_j(t) - \mathscr{S}_j(t)\right\} \leq -\frac{g(0)\alpha}{16}\mathscr{M}_j(t);$$

$$\left|\mathscr{S}_j(t)\right| < \frac{c_0}{2}\mathscr{M}_j(t).$$

So, using (2.22) we have

$$\frac{c_0}{2}\mathscr{M}_j(t) \leq \mathscr{L}_j(t) - \mathscr{S}_j(t) \leq 2c_1\mathscr{M}_j(t), \tag{2.27}$$

and therefore we get

$$\frac{d}{dt}\left\{\mathscr{L}_j(t) - \mathscr{S}_j(t)\right\} \leq -\frac{g(0)\alpha}{32c_1}\left\{\mathscr{L}_j(t) - \mathscr{S}_j(t)\right\},$$

which implies that

$$\mathscr{L}_j(t) - \mathscr{S}_j(t) \leq \left\{\mathscr{L}_j(0) - \mathscr{S}_j(0)\right\}e^{-\gamma t},$$

where $\gamma = g(0)\alpha/32c_1$. The above inequality together with (2.27) yields

$$\mathscr{M}_j(t) \leq \frac{2}{c_0}\left\{\mathscr{L}_j(0) - \mathscr{S}_j(0)\right\}e^{-\gamma t} \leq \frac{4c_1}{c_0}\mathscr{M}_j(0)e^{-\gamma t}, \qquad \forall t \in [0, T^*[$$

$$\begin{aligned}\left\{|h_j''|^2 + \lambda_j|h_j'|^2 + \lambda_j^2 g \,\square\, h_j + \lambda_j^2 h_j^2\right\}&\\ \leq \frac{4c_1}{c_0}\left\{\left|h_j''(0)\right|^2 + \lambda_j\left|h_j'(0)\right|^2 + \lambda_j^2\left|h_j(0)\right|^2\right\}e^{-\gamma t}.&\end{aligned} \tag{2.28}$$

Our next step is to show that $T_{\max} = \infty$. To do this we will reason by contradiction. Let us suppose that $T^* < T_{\max} < \infty$ and also suppose that $T^* = T_1^*$. Multiplying (2.28) by λ_j^{-1} and summing up in j, we have

$$\begin{aligned}\mathscr{F}(t) &= \left\{\left\|u_{tt}(t)\right\|_{V'}^2 + \left\|u_t(t)\right\|^2 + \|A^{1/2}u\|^2 + g \,\square\, A^{1/2}u\right\}\\ &\leq \frac{4c_1}{c_0}\mathscr{F}(0)e^{-\gamma t},\end{aligned} \tag{2.29}$$

where $V = D(A^{1/2})$. On the other hand, multiplying (2.21) by λ_j^{-1} and summing up in j we get

$$\|u_{tt}(t)\|_{V'}^2 \leq 17\big(\|A^{1/2}u\|^2 + g \,\square\, A^{1/2}u\big).$$

So, we have

$$E_1(t) \leq \mathscr{F}(t) \leq 18\Big\{\|u_t(t)\|^2 + \|A^{1/2}u\|^2 + g \,\square\, A^{1/2}u\Big\} = 18E_1(t).$$

Therefore from (2.29) we get

$$E_1(t) \leq \frac{4c_1}{c_0}\mathscr{F}(0)e^{-\gamma t} \leq \frac{4c_1}{c_0}18E_1(0)e^{-\gamma t} < d\varepsilon\, e^{-\gamma t};$$

letting $t \to T^* = T_1^*$ we get

$$E_1(T_1^*) \leq de^{-\gamma t_1^*}\,\varepsilon < d\varepsilon$$

which is contradictory to the maximality of T_1^*. Suppose now that $T^* = T_2^*$. Similarly, summing up (2.28) in j we have

$$\mathscr{G}(t) = \Big\{\|u_{tt}(t)\|^2 + \|A^{1/2}u_t\|^2 + g \,\square\, Au + \|Au(t)\|^2\Big\} \leq \frac{4c_1}{c_0}\mathscr{G}(0)e^{-\gamma t}.$$

Since $\mathscr{G}(t) > E_2(t)$ and $\mathscr{G}(0) = E_2(0)$, we get

$$E_2(t) < \mathscr{G}(t) \leq \frac{4c_1}{c_0}\mathscr{G}(0)e^{-\gamma t} = \frac{4c_1}{c_0}E_2(0)e^{-\gamma t} < d\mu\, e^{-\gamma t}.$$

Letting $t \to T^* = T_2^*$ we get

$$E_2(T_2^*) \leq d\mu\, e^{-\gamma T_2^*} < d\mu;$$

which is also contradictory to the maximality of T_2^*. Hence, $T_{\max} = \infty$. To prove the uniqueness, let us suppose that u^1 and u^2 are two solutions of (2.1) denoted by $U = u^1 - u^2$. Then we have that U satisfies

$$\begin{aligned} U_{tt} + M([u^1])AU = {} & \{M([u^1]) - M([u^2])\}Au^2 \\ & + g * \{N([u^1]) - N([u^2])\}Au^1 + g * N([u^2])AU \\ & U(0) = U_t(0) = 0. \end{aligned}$$

Multiplying the above inequality by U_t we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|U_t(t)\|^2 + M([u^1])\|A^{1/2}U(t)\|^2\right\} \\
&\quad= \frac{1}{2}\left(\frac{d}{dt}M([u^1])\right)\|A^{1/2}U\|^2 + \left\{M([u^1]) - M([u^2])\right\}(Au^2, U_t) \\
&\qquad + g*\left\{N([u^1]) - N([u^2])\right\}(Au^1, U_t) + \left(g*\left\{N([u^2])AU\right\}, U_t\right) \\
&\quad= \frac{1}{2}\left(\frac{d}{dt}M([u^1])\right)\|A^{1/2}U\|^2 + \left\{M([u^1]) - M([u^2])\right\}(Au^2, U_t) \\
&\qquad + g*\left\{M([u^1]) - M([u^2])\right\}(Au^1, U_t) \\
&\qquad + \frac{d}{dt}\left(g*\left\{N([u^2])AU\right\}, U\right) - g(0)\left(N([u^2])AU, U\right) \\
&\qquad - \left(g'*\left\{N([u^2])AU\right\}, U\right).
\end{aligned} \tag{2.30}$$

Denoting

$$\mathscr{K}(t) = \|U_t(t)\|^2 + M([u^1])\|A^{1/2}U(t)\|^2 - 2\left(g*N([u^2])AU, U\right)$$

and using the mean value theorem, from (2.30) we get that there exists a positive constant C for which we have

$$\frac{d}{dt}\mathscr{K}(t) \leq C\left\{\|U_t(t)\|^2 + \|A^{1/2}U(t)\|^2 + \int_0^t\|A^{1/2}U(\tau)\|^2\,d\tau\right\}.$$

Integrating over $[0, t]$ we get

$$\mathscr{K}(t) \leq C(1+t)\int_0^1\|U_t(\tau)\|^2 + \|A^{1/2}U(\tau)\|^2\,d\tau.$$

Since

$$\begin{aligned}
\left|\left(g*N([u^2])AU, U\right)\right| &= \left|\left(g*N([u^2])A^{1/2}U, A^{1/2}U\right)\right| \\
&\leq C_0\int_0^t\|A^{1/2}U(\tau)\|^2\,d\tau + \frac{m_0}{2}\|A^{1/2}U(t)\|^2,
\end{aligned}$$

where m_0 is such that $m_0 \leq M([u^1])$ and C_0 is a positive constant, so we have

$$\|U_t(t)\|^2 + \frac{m_0}{2}\|A^{1/2}U(t)\|^2 \leq C_1(1+t)\int_0^t\|U_t(\tau)\|^2 + \|A^{1/2}U(\tau)\|^2\,d\tau.$$

Using Gronwall's inequality the uniqueness follows. The proof is now complete. ∎

Remark 2.1. We can establish a similar existence result for Eq. (1.4), that is, when $S(0) := S_0 > 0$. In this case instead of (2.5), we use

$$h_j'' + \lambda_j h_j - \lambda_j \int_0^t g(t-\tau) h_j(\tau)\, d\tau + S_0 \lambda_j^\beta h_j' = p_j := q_j + r_j + t_j, \quad (2.31)$$

where q_j and r_j are as before and

$$t_j := \{S(0) - S([u])\} \lambda_j^\beta h_j'.$$

Multiplying Eq. (2.31) by h_j' we get

$$\frac{1}{2}\frac{d}{dt} \underbrace{\left\{ |h_j'|^2 + \lambda_j \left(1 - \int_0^t g\, d\tau\right) |h_j|^2 + \lambda_j g \,\Box\, h_j \right\}}_{:= \hat{E}_j(t)} \leq -\frac{\lambda_j}{2} g |h_j|^2 - \frac{c_{\lambda j}}{2} g \,\Box\, h_j + p_j h_j' - S_0 \lambda_j^\beta |h_j'|^2.$$

Multiplying by h_j we get

$$\frac{d}{dt} \underbrace{\left\{ h_j h_j' + \frac{S_0}{2} \lambda_j^\beta |h_j|^2 \right\}}_{:= \hat{I}_j(t)} \leq |h_j'|^2 - \frac{\lambda_j \alpha}{2} |h_j|^2 + \frac{\lambda_j}{2\alpha} g \,\Box\, h_j + p_j h_j.$$

Now define the functionals

$$\hat{\mathscr{L}}_j(t) = N \hat{E}_j^1(t) + \hat{I}_j(t)$$

$$\hat{\mathscr{M}}_j(t) = |h_j'|^2 + \lambda_j |h_j|^2 + \lambda_j g \,\Box\, h_j$$

$$E_1(t) = \|u_t\|^2 + \|A^{1/2} u\|^2 + g \,\Box\, A^{1/2} u$$

$$E_2(t) = \|u_{tt}\|^2 + \|A^\theta u_t\|^2 + \|A^{\theta + 1/2} u\|^2,$$

where N is chosen such that there exist positive constants c_0 and c_1 satisfying

$$c_0 \hat{\mathscr{M}}_j(t) \leq \hat{\mathscr{L}}_j(t) \leq c_1 \hat{\mathscr{M}}_j(t). \quad (2.32)$$

Using the same reasoning as in Theorem 2.2 we can establish the following result

THEOREM 2.3. *Under the same hypotheses as Lemma* 2.1, *there exists* $\varepsilon > 0$ *such that for any*

$$(u_0, u_1) \in D(A) \times D(A^\theta)$$

satisfying

$$\|A^{1/2}u_0\|^2 + \|u_1\|^2 < \varepsilon,$$

there exists only one solution u of Eq. (2.1) *such that*

$$u \in C^2([0,\infty[; H) \cap C^1([0,\infty[; D(A^\theta)) \cap C([0,\infty[; D(A)),$$

which decays as

$$E_1(t) \leq E_1(0)e^{-\gamma t} \qquad \forall t \geq 0 \quad \textit{and} \quad \gamma > 0.$$

3. SMOOTHNESS EFFECT

In this section we prove that the solution of the equation

$$u_{tt} + M([u])A^2u + S([u])A^\beta u_t + 0 \tag{3.1}$$
$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

has the smoothing effect property on the initial data provided $0 < \beta < 2$ and $S(0) > 0$. That is, if the initial data belong to $D(A^{2\theta+1}) \times D(A^{2\theta})$, where $\theta = \max\{1/2, \beta/2\}$ then the solution $u \in C^1(]0,T], D(A^k))$, $\forall k \in \mathbb{N}$. To do this, note that the continuity of u and the hypotheses on M and S imply

$$m_1 \geq m(t) := M([u(t)]) \geq m_0 > 0 \qquad \forall t \in [0,T]$$
$$s_1 \geq s(t) := S([u(t)]) \geq s_0 > 0 \tag{3.2}$$
$$m \in C^1([0,T]; \mathbb{R}) \Rightarrow |m'(t)| \leq c \qquad \forall t \in [0,T], \tag{3.3}$$

where m_0, m_1, s_0, s_1, and c are positive constants. So, the main result of this section is:

THEOREM 3.1. *Let us suppose that* $(u_0, u_1) \in D(A^{2\theta+1}) \times D(A^{2\theta})$ *and u is a solution of* (3.1). *Then for any* $t \in [\delta, T]$, $\delta > 0$, *and* $\forall k \in \mathbb{N}$ *we have*

$$u(t) \in D(A^k) \qquad \textit{and} \qquad u_t(t) \in D(A^k),$$

that is, $u \in C^1(]0,T[; D(A^\infty))$ *where* $D(A^\infty) = \bigcap_{k\in\mathbb{N}} D(A^k)$.

Proof. We will show that

$$\sum_{i=1}^{\infty} \lambda_i^m h_i^2(t) \qquad \text{and} \qquad \sum_{i=1}^{\infty} \lambda_i^l h_i'(t)^2$$

are convergent for any $t \in [\delta, T]$, $\delta > 0$, and $\forall m, l \in \mathbb{N}$. To do this, let us project Eq. (3.1) over $\mathbb{R}w_i$ to get

$$h_i''(t) + m(t)\lambda_i^2 h_i(t) + s(t)\lambda_i^\beta h_i'(t) = 0. \tag{3.4}$$

Multiplying (3.4) by $h_i'(t)$ and using (3.3) and (3.2) we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{h_i'(t)^2 + m(t)\lambda_i^2 h_i(t)^2\right\} &= \frac{m'(t)}{2}\lambda_i^2 h_i^2(t) - s(t)\lambda_i^\beta h_i'(t)^2 \\ &\leq \frac{c}{2}\lambda_i^2 h_i^2(t) - s_0\lambda_i^\beta h_i'(t)^2. \end{aligned} \tag{3.5}$$

To show the smoothing effect we consider two cases, first when $\beta \in]0,1]$ and then $\beta \in]1,2[$. Let us suppose that $0 < \beta \leq 1$. From (3.4) and (3.2) we have

$$\begin{aligned} \frac{d}{dt}\left\{\lambda_i^\beta h_i(t) h_i'(t)\right\} &= \lambda_i^\beta h_i'(t)^2 - m(t)\lambda_i^{2+\beta} h_i(t)^2 - s(t)\lambda_i^{2\beta} h_i'(t) h_i(t) \\ &\leq \lambda_i^\beta h_i'(t)^2 - m_0\lambda_i^{2+\beta} h_i(t)^2 \\ &\quad + \sqrt{m_0}\,\lambda_i^{(2+\beta)/2}|h_i(t)|\frac{\lambda_i^{(3\beta-2)/2}}{\sqrt{m_0}}|h_i'(t)|s_1 \\ &\leq \left(1 + \frac{s_1^2}{2m_0}\right)\lambda_i^\beta h_i'(t)^2 - \frac{m_0}{2}\lambda_i^{2+\beta} h_i(t)^2 \end{aligned} \tag{3.6}$$

Let us introduce the functionals

$$\mathscr{P}_i(t) = \frac{N_0}{2}h_i'(t)^2 + \frac{N_0}{2}m(t)\lambda_i^2 h_i(t)^2 + \lambda_i^\beta h_i(t) h_i'(t)$$

$$\mathscr{Q}_i(t) = h_i'(t)^2 + \lambda_i^2 h_i^2(t),$$

where N_0 is such that $N_0 > \max\{(2m_0 + s_1^2)/(2m_0 s_0), 1, 1/m_0\}$. From (3.5), (3.6), and our choice of N_0, there exist positive constants α_0, c_1, and c_2 such that

$$\frac{d}{dt}\mathscr{P}_i(t) \leq -\alpha_0\lambda_i^\beta h_i'(t)^2 - \frac{\lambda_i^2 m_0}{2}\left(\lambda_i^\beta - \frac{cN_0}{m_0}\right)h_i^2(t); \tag{3.7}$$

$$c_2\mathscr{Q}_i(t) \leq \mathscr{P}_i(t) \leq c_1\mathscr{Q}_i(t). \tag{3.8}$$

Since $\lambda_i \to \infty$ when $i \to \infty$, there exists $I \in \mathbb{N}$ such that

$$\forall i \geq I \Rightarrow \lambda_i^{\beta} - \frac{cN_0}{m_0} \geq \frac{\lambda_i^{\beta}}{2}. \tag{3.9}$$

From (3.7) and (3.9) we get

$$\frac{d}{dt}\mathscr{P}_i(t) \leq -c_0 \lambda_i^{\beta}\left\{h_i'(t)^2 + \lambda_i^2 h_i(t)^2\right\} \leq -c_0 \lambda_i^{\beta}\mathscr{Q}_i(t) \qquad \forall i \geq I, \tag{3.10}$$

where $c_0 = \min\{\alpha_0, m_0/4\}$. From (3.7) and (3.10) it follows that

$$\mathscr{Q}_i(t) \leq \frac{c_1}{c_2}\mathscr{Q}_i(0)e^{-\gamma\lambda_i^{\beta}t} \qquad \forall i \geq I, \quad \text{and} \quad t \in [\delta, T],$$

where $\gamma = c_0/c_1$. Multiplying the above inequality by $\lambda_i^{m\beta}$ and keeping in mind the definition of $\mathscr{Q}$ and that

$$\lambda_i^{m\beta}e^{-\lambda_i^{\beta}\gamma\delta} \leq \frac{m!}{(\gamma\delta)^m} =: c_m(\delta),$$

we get

$$\lambda_i^{m\beta}h_i'(t)^2 + \lambda_i^{m\beta+2}h_i^2(t) \leq \frac{c_1 c_m(\delta)}{c_2}\left\{h_i'(0)^2 + \lambda_i^2 h_i(0)^2\right\}. \tag{3.11}$$

Summing up (3.11) from $i = I$ to N we have

$$\begin{aligned}\left\{\sum_{i=I}^{N}\lambda_i^{m\beta}h_i'(t)^2 + \sum_{i=I}^{N}\lambda_i^{m\beta+2}h_i(t)^2\right\} &\leq \frac{c_1 c_m(\delta)}{c_2}\sum_{i=I}^{N}\left\{h_i'(0)^2 + \lambda_i^2 h_i(0)^2\right\}\\ &\leq \frac{c_1 c_m(\delta)}{c_2}\left\{\|u_1\|^2 + \|Au_0\|^2\right\}.\end{aligned}$$

Since the right hand side does not depend on N the series

$$\sum_{i=1}^{\infty}\lambda_i^{m\beta}h_i'(t)^2 \qquad \text{and} \qquad \sum_{i=1}^{\infty}\lambda_i^{m\beta+2}h_i(t)^2$$

are convergent to any $t \in [\delta, T]$ and any $m \in \mathbb{N}$. Therefore, $u(t) \in D(A^{(m+2)/2})$ and $u_t(t) \in D(A^{m/2})$, $\forall m \in \mathbb{N}$ and $\forall t \in [\delta, T]$. From this our result follows. Now let us consider the case $1 < \beta < 2$. From (3.4) and (3.2) we have

$$\frac{d}{dt}\left\{\lambda_i^{2-\beta}h_i(t)h_i'(t)\right\} \leq \left(1 + \frac{s_i^2}{2m_0}\right)\lambda_i^{\beta}h_i'(t)^2 - \frac{m_0}{2}\lambda_i^{4-\beta}h_i(t)^2. \tag{3.12}$$

Let us consider

$$\mathscr{K}_i(t) := \frac{N_0}{2} h_i'(t)^2 + \frac{N_0}{2} m(t) \lambda_i^2 h_i(t)^2 + \lambda_i^{2-\beta} h_i(t) h_i'(t),$$

where

$$N_0 > \max\left\{ \frac{2m_0 + s_1^2}{2m_0 s_0}, 1, \frac{1}{m_0} \right\}.$$

From (3.5), (3.12), and our choice of N_0 there exists positive constants α_1, c_1, and c_2 such that

$$\frac{d}{dt}\mathscr{K}_i(t) \leq -\alpha_1 \lambda_1^{\beta} h_i'(t)^2 - \left(\lambda_i^{2-\beta} - \frac{N_0 c}{m_0} \right) \frac{m_0}{2} \lambda_i^2 h_i(t)^2, \quad (3.13)$$

$$c_2 \mathscr{Q}_i(t) \leq \mathscr{K}_i(t) \leq c_1 \mathscr{Q}_i(t), \quad (3.14)$$

where

$$\mathscr{Q}_i(t) = h_i'(t)^2 + \lambda_i^2 h_i(t)^2.$$

Since $\lambda_i \to \infty$ as $i \to \infty$, there exists $I \in \mathbb{N}$ such that

$$\lambda_i^{2-\beta} - \frac{cN_0}{m_0} \geq \frac{1}{2} \lambda_i^{2-\beta}, \qquad \forall i \geq I. \quad (3.15)$$

From (3.13), (3.15) we have

$$\frac{d}{dt}\mathscr{K}_i(t) \leq -c_3 \lambda_i^{2-\beta} \mathscr{Q}_i(t), \qquad \forall i \geq I, \quad (3.16)$$

where $c_3 = \min\{\alpha_1, m_0/4\}$. From (3.14) and (3.16) it follows that

$$\mathscr{Q}_i(t) \leq \frac{c_1}{c_2} \mathscr{Q}_i(0) e^{-\gamma \lambda_i^{2-\beta} t},$$

for $\gamma = c_3/c_1$. Therefore we et

$$\begin{aligned} &\lambda_i^{(2-\beta)m} h_i'(t)^2 + \lambda_i^{(2-\beta)m+2} h_i(t)^2 \\ &\quad \leq \frac{c_1}{c_2} \left\{ h_i'(0)^2 + \lambda_i^2 h_i(0)^2 \right\} e^{-\gamma \lambda_i^{2-\beta} \delta} \lambda_i^{(2-\beta)m} \\ &\quad \leq \frac{c_1 c_m(\delta)}{c_2} \left\{ h_i'(0)^2 + \lambda_i^2 h_i^2(0) \right\} \end{aligned}$$

from which it follows that

$$\sum_{i=I}^{\infty} \lambda_i^{(2-\beta)m} h_i'(t)^2 + \sum_{i=I}^{\infty} \lambda_i^{(2-\beta)m+2} h_i(t)^2 \leq \frac{c_1 c_m(\delta)}{c_2} \left\{ \|u_1\|^2 + \|Au_0\|^2 \right\},$$

for any $t \in [\delta, T]$ and $m \in \mathbb{N}$. So, we have

$$u(t) \in D(A^{(m+2)/2}) \qquad \text{and} \qquad u_t(t) \in D(A^{m/2}),$$

for any $m \in \mathbb{N}$ and $t \in [\delta, T]$. The proof is now complete. ∎

4. PROPAGATION OF SINGULARITIES

In this section we consider the memory effect on Eq. (1.4) and we show that this effect is stronger than the viscoelastic damping, in the sense that it destroys the regularity effect showed in the above section, making the resulting equation have an hyperbolic behaviour. We consider the equation

$$u_{tt} + A^2 u - \int_0^t g(t-\tau) A^2 u(\tau)\, d\tau + Au_t = 0 \tag{4.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),$$

where $g(t) = e^{-\gamma t}$ for $\gamma \neq 0$.

Remark 4.1. Since A is an abstract operator the existence result of Section 2 can also applied to Eq. (4.1). Here we write A^2 instead of A to be more emphatic on the applications of the plate equations.

The spectral equation associated to (4.1) is given by

$$h_j''(t) + \lambda_j^2 h_j(t) - \lambda_j^2 \int_0^t g(t-\tau) h_j(\tau)\, d\tau + \lambda_j h_j'(t) = 0. \tag{4.2}$$

Differentiating Eq. (4.2) with respect to t yields

$$h_j'''(t) + \lambda_j^2 h_j'(t) - \lambda_j^2 g(0) h_j(t) - \lambda_j^2 \int_0^t g'(t-\tau) h_j(\tau)\, d\tau + \lambda_j h_j''(t) = 0.$$

Using (4.2) and since $g(0) = 1$, $g'(t) = -\gamma g(t)$ we get

$$h_j'''(t) + (\gamma + \lambda_j) h_j''(t) + \left(\lambda_j^2 + \gamma\lambda_j\right) h_j'(t) + (\gamma - 1)\lambda_j^2 h_j(t) = 0. \tag{4.3}$$

The general solution of Eq. (4.3) is of the form

$$h_j(t) = c_{1j} e^{r_{1j} t} + c_{2j} e^{r_{2j} t} + c_{3j} e^{r_{3j} t},$$

where r_{1j}, r_{2j}, and r_{3j} are the roots of the characteristic polynomial

$$r_j^3 + (\gamma + \lambda_j) r_j^2 + \left(\lambda_j^2 + \gamma\lambda_j\right) r_j + (\gamma - 1)\lambda_j^2 = 0. \tag{4.4}$$

Note that any solution h_j of (4.3) can be written as

$$h_j(t) = h_j^1(t) + h_j^2(t) + h_j^3(t),$$

where

$h_j^1(t)$ is the solution of (4.3) for $h_j'(0) = h_j''(0) = 0$;

$h_j^2(t)$ is the solution of (4.3) for $h_j(0) = h_j''(0) = 0$;

$h_j^3(t)$ is the solution of (4.3) for $h_j(0) = h_j'(0) = 0$.

Observe that $h_j(t)$ is also a solution of (4.2) if

$$h_j''(0) + \lambda_j^2 h_j(0) + \lambda_j h_j'(0) = 0.$$

First we consider the case $h_j'(0) = 0$ and $h_j''(0) = -\lambda_j^2 h_j(0)$. Since

$$h_j^1(t) = h_j(0)\left\{d_{1j}^1 e^{r_{1j}t} + d_{2j}^1 e^{r_{2j}t} + d_{3j}^1 e^{r_{3j}t}\right\},$$

$$h_j^3(t) = h_j''(0)\left\{d_{1j}^3 e^{r_{1j}t} + d_{2j}^3 e^{r_{2j}t} + d_{3j}^3 e^{r_{3j}t}\right\},$$

where

$$d_{1j}^1 = \frac{r_{2j}r_{3j}}{(r_{3j} - r_{1j})(r_{2j} - r_{1j})}, \qquad d_{2j}^1 = \frac{r_{1j}r_{3j}}{(r_{2j} - r_{1j})(r_{2j} - r_{3j})},$$

$$d_{3j}^1 = \frac{r_{2j}r_{1j}}{(r_{2j} - r_{3j})(r_{1j} - r_{3j})}, \tag{4.5}$$

$$d_{1j}^3 = \frac{1}{(r_{2j} - r_{1j})(r_{3j} - r_{1j})}, \qquad d_{2j}^3 = \frac{1}{(r_{2j} - r_{3j})(r_{2j} - r_{1j})},$$

$$d_{3j}^3 = \frac{1}{(r_{3j} - r_{2j})(r_{3j} - r_{1j})}, \tag{4.6}$$

it follows that

$$h_j(t) = h_j(0)\left\{d_{1j} e^{r_{1j}t} + d_{2j} e^{r_{2j}t} + d_{3j} e^{r_{3j}t}\right\} \tag{4.7}$$

where

$$d_{ij} = d_{ij}^1 - \lambda_j^2 d_{ij}^3, \qquad i = 1, 2, 3. \tag{4.8}$$

Finally, we consider the case $h_j(0) = 0$, $h''_j(0) = -\lambda_j h'_j(0)$. So we have that

$$h_j(t) = h'_j(0)\left\{\hat{d}_{1j} e^{r_{1j}t} + \hat{d}_{2j} e^{r_{2j}t} + \hat{d}_{3j} e^{r_{3j}t}\right\}, \tag{4.9}$$

where

$$\hat{d}_{ij} = d_{ij}^2 - \lambda_j e_{ij}^3, \qquad i = 1,2,3, \tag{4.10}$$

and

$$d_{1j}^2 = \frac{r_{2j} + r_{3j}}{(r_{1j} - r_{3j})(r_{2j} - r_{1j})}, \qquad d_{2j}^2 = \frac{r_{3j} + r_{1j}}{(r_{2j} - r_{1j})(r_{3j} - r_{2j})},$$
$$d_{3j}^2 = \frac{r_{1j} + r_{2j}}{(r_{1j} - r_{3j})(r_{3j} - r_{2j})}, \tag{4.11}$$

with

$$\begin{aligned} C_j &= 9\lambda_j^6 + (-18\gamma + 42)\lambda_j^5 + \left(27\gamma^2 - 90\gamma + 81\right)\lambda_j^4 \\ &\quad + \left(-18\gamma^3 + 18\gamma^2\right)\lambda_j^3 + \left(9\gamma^4 - 12\gamma^3\right)\lambda_j^2; \\ A_j &= \frac{\lambda_j^2(-5\gamma + 9)}{18} + \frac{7\lambda_j^3}{54} + \frac{\lambda_j\gamma^2}{18} - \frac{\gamma^3}{27} + \frac{1}{18}\sqrt{C_j}; \\ B_j &= \frac{(1/9)\left(2\lambda_j^2 + \gamma\lambda_j - \gamma^2\right)}{\sqrt[3]{A_j}}. \end{aligned}$$

The roots of (4.4) are given by

$$\begin{aligned} r_{1j} &= A_j^{1/3} - B_j - \frac{1}{3}(\gamma + \lambda_j); \\ r_{2j} &= -\frac{1}{2}\left(A_j^{1/3} - B_j\right) - \frac{1}{3}(\gamma + \lambda_j) + \frac{i\sqrt{3}}{2}\left\{A_j^{1/3} + B_j\right\}; \\ r_{3j} &= -\frac{1}{2}\left(A_j^{1/3} - B_j\right) - \frac{1}{3}(\gamma + \lambda_j) - \frac{i\sqrt{3}}{2}\left\{A_j^{1/3} + B_j\right\}. \end{aligned} \tag{4.12}$$

To show the propagation of singularities we need to know the behavior of the roots at infinity. This behaviour is summarized in the following lemma.

LEMMA 4.1. *When $j \to \infty$ we have*

$$r_{1j} = (1-\gamma) + O\left(\frac{1}{\lambda_j}\right)$$

$$r_{2j} = -\frac{\lambda_j}{2} - \frac{1}{2} + O\left(\frac{1}{\lambda_j}\right) + \frac{i\sqrt{3}}{2}\left\{\lambda_j + \frac{1}{3} + O\left(\frac{1}{\lambda_j}\right)\right\}$$

$$r_{3j} = -\frac{\lambda_j}{2} - \frac{1}{2} + O\left(\frac{1}{\lambda_j}\right) - \frac{i\sqrt{3}}{2}\left\{\lambda_j + \frac{1}{3} + O\left(\frac{1}{\lambda_j}\right)\right\}.$$

Proof. Recalling the definition of C_j we have

$$C_j^{1/2} = 3\lambda_j^3\left\{1 + \frac{2(7-3\gamma)}{3\lambda_j} + \frac{(3\gamma^2 - 10\gamma + 9)}{\lambda_j^2} + \frac{2\gamma^2(1-\gamma)}{\lambda_j^3} + \frac{\gamma^3(3\gamma-4)}{3\lambda_j^4}\right\}^{1/2}.$$

Since $(1+x)^\alpha = 1 + \alpha x + O(x^2)$, $x \to 0$, $\alpha \in \mathbb{R}$ then

$$C_j^{1/2} = 3\lambda_j^3\left\{1 + \frac{(7-3\gamma)}{3\lambda_j} + \frac{(3\gamma^2 + 10\gamma + 9)}{2\lambda_j^2} + \frac{\gamma^2(1-\gamma)}{\lambda_j^3} + \frac{\gamma^3(3\gamma-4)}{6\lambda_j^4} + O\left(\frac{1}{\lambda_j^2}\right)\right\},$$

that is,

$$C_j^{1/2} = 3\lambda_j^3\left\{1 + \frac{(7-3\gamma)}{\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right\}.$$

From this it follows that A_j and B_j satisfy

$$\begin{aligned}
A_j^{1/3} &= \left[\frac{\lambda_j^2(-5\gamma + 9)}{18} + \frac{7\lambda_j^3}{54} + \frac{\gamma^2\lambda_j}{18} - \frac{\gamma^3}{27} \right. \\
&\qquad \left. + \frac{\lambda^3}{6}\left\{1 + \frac{(7-3\gamma)}{3\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right\}\right]^{1/3} \\
&= \left[\frac{4\lambda_j^2(-\gamma + 2)}{9} + \frac{8\lambda_j^3}{27} + \frac{\gamma^2\lambda_j}{18} - \frac{\gamma^3}{27} + O(\lambda_j) \right]^{1/3} \\
&= \frac{2\lambda_j}{3}\left\{1 + \frac{(2-\gamma)}{2\lambda_j} + \frac{\gamma^2}{17\lambda_j^2} - \frac{\gamma^3}{24\lambda_j^3} + O\left(\frac{1}{\lambda_j^2}\right)\right\} \\
&= \frac{2\lambda_j}{3} + \frac{(2-\gamma)}{3} + \frac{\gamma^2}{24\lambda_j} - \frac{\gamma^3}{36\lambda_j^2} + O\left(\frac{1}{\lambda_j}\right), \\
B_j &= \frac{1}{3}\lambda_j + \frac{1}{3}(\gamma - 1) + O\left(\frac{1}{\lambda_j}\right).
\end{aligned}$$

From this we have

$$\begin{aligned}
A_j^{1/3} - B_j &= \frac{\lambda_j}{3} + \frac{(3-2\gamma)}{3} + \frac{\gamma^2}{24\lambda_j} - \frac{\gamma^3}{36\lambda_j^2} + O\left(\frac{1}{\lambda_j}\right) \\
&= \frac{(3-2\gamma)}{3} + \frac{\lambda_j}{3} + O\left(\frac{1}{\lambda_j}\right) \\
A_j^{1/3} + B_j &= \lambda_j + \frac{1}{3} + \frac{\gamma^2}{24\lambda_j} - \frac{\gamma^3}{36\lambda_j^2} + O\left(\frac{1}{\lambda_j}\right) \\
&= \lambda_j + \frac{1}{3} + O\left(\frac{1}{\lambda_j}\right).
\end{aligned}$$

Using (4.12) our result follows. ∎

As a consequence of Lemma 4.1, the asymptotic behaviour of the coefficients d_{1j}, d_{2j}, d_{3j}, $\hat{d}_{1j}$, $\hat{d}_{2j}$, and $\hat{d}_{3j}$ follows, which we establish in the following lemma,

LEMMA 4.2. *When $j \to \infty$ we have*

$$d_{1j} = \frac{1}{2\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)$$

$$d_{2j} = \frac{1}{2} + O\left(\frac{1}{\lambda_j}\right) + i\left\{\frac{\sqrt{3}}{6} + O\left(\frac{1}{\lambda_j}\right)\right\}$$

$$d_{3j} = \frac{1}{2} + O\left(\frac{1}{\lambda_j}\right) - i\left\{\frac{\sqrt{3}}{6} + O\left(\frac{1}{\lambda_j}\right)\right\}$$

$$\hat{d}_{1j} = \frac{1}{\lambda_j^2} + O\left(\frac{1}{\lambda_j^3}\right)$$

$$\hat{d}_{2j} = O\left(\frac{1}{\lambda_j^2}\right) - i\left\{\frac{\sqrt{3}}{3\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right\}$$

$$\hat{d}_{3j} = O\left(\frac{1}{\lambda_j^2}\right) + i\left\{\frac{\sqrt{3}}{3\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right\}.$$

Proof. The result follows immediately from Lemma 4.1 and relations (4.5)–(4.6) and (4.11). ▮

THEOREM 4.1. *Let us denote by u the solution of* (4.1) *for initial data such that* $(u_0, u_1) \in D(A^{\rho}) \times D(A^{\rho-1})$ *for* $p \geq 2$ *but* $(u_0, u_1) \notin D(A^{\rho+s}) \times D(A^{\rho+s-1})$, $\forall s > 0$. *Then* $u(t) \notin D(A^{\rho+1+s})$, $\forall s > 0$, $\forall t \in [0, T]$. *More precisely, the solution u can be decomposed into two parts*; *one of them propagates singularities and the other contains the smoothing effect property*.

Proof. The solution u of (4.1) is given by

$$u(t) = \sum_{j=1}^{\infty} h_j(t) w_j.$$

First we will suppose that $u_0 \neq 0$ and $u_1 = 0$. Then we will prove the theorem for the case $u_0 = 0$ and $u_1 \neq 0$. Our conclusion will follow using superposition. From (4.7) we have

$$\begin{aligned} u(t) &= \sum_{j=1}^{\infty} w_j h_j(0) d_{1j} e^{r_{1j}t} + \sum_{j=1}^{\infty} w_j h_j(0)\left\{d_{2j} e^{r_{2j}t} + d_{3j} e^{r_{3j}t}\right\} \\ &\equiv: u_1(t) + u_2(t). \end{aligned}$$

We will show that u_2 has the smoothness effect on the initial data, while u_1 propagates singularities. From Lemmas 4.1 and 4.2, taking j_0 large enough, we have $\forall m \in \mathbb{N}$ and $\forall t \geq \delta > 0$ that

$$\sum_{j=j_0}^{\infty} \lambda_j^{2m} \left| h_j(0) d_{2j} e^{r_{2j}t} \right|^2 \leq c \sum_{j=j_0}^{\infty} \lambda_j^{2m} h_j^2(0) e^{-\lambda_j t} = c \sum_{j=j_0}^{\infty} \lambda_j^2 h_j^2(0) \lambda_j^{2m-2} e^{-\lambda_j t}.$$

Since

$$\frac{(\lambda_j t)^{2m-2}}{(2m-1)!} \leq e^{\lambda_j t} \Rightarrow \lambda_j^{2m-2} e^{-\lambda_j t} \leq \frac{(2m-2)!}{t^{2m-2}} \leq \frac{(2m-2)!}{\delta^{2m-2}}$$

then

$$\sum_{j=j_0}^{\infty} \lambda_j^{2m} \left| h_j(0) d_{2j} e^{r_{2j}t} \right|^2 \leq \frac{c(2m-2)!}{\delta^{2m-2}} \sum_{j=j_0}^{\infty} \lambda_j^2 h_j^2(0) < \infty$$

because $u_0 \in D(A)$. Similarly, we can prove that for any $m \in \mathbb{N}$

$$\sum_{j=1}^{\infty} \lambda_j^{2m} \left| h_j(0) d_{2j} r_{2j} e^{r_{2j}t} \right|^2 < +\infty$$

$$\sum_{j=1}^{\infty} \lambda_j^{2m} \left| h_j(0) d_{3j} e^{r_{3j}t} \right|^2 < +\infty$$

$$\sum_{j=1}^{\infty} \lambda_j^{2m} \left| h_j(0) d_{3j} r_{3j} e^{r_{3j}t} \right|^2 < +\infty.$$

Then $u_2 \in C^1(]0, T], D(A^\infty))$. Finally, we will prove that $u(t) \notin D(A^{\rho+1+s})$. To do this we first take $u_1 = 0$ and $u_0 \in D(A^\rho)$ such that $u_0 \notin D(A^{\rho+s})$. In fact, let us suppose the contrary, that is,

$$\sum_{j=1}^{\infty} \lambda_j^{2(\rho+1+s)} h_j^2(0) d_{1j}^2 e^{2r_{1j}t} < +\infty.$$

From Lemma 4.1 and 4.2 there exist positive constants c_1, c_2 and $j_0 \in \mathbb{N}$ such that if $j > j_0 \Rightarrow \lambda_j d_{1j} > c_1$ and $r_{1j} > c_2$, then

$$+\infty > \sum_{j=j_0}^{\infty} \lambda_j^{2(\rho+s+1)} h_j^2(0) d_{1j}^2 e^{2r_{1j}t} > c_1^2 e^{2c_2 t} \sum_{j=j_0}^{\infty} \lambda_j^{2(\rho+s)} h_j^2(0),$$

which is a contradiction because $u_0 \notin D(A^{\rho+s})$. To prove the case $u_0 = 0$ and $u_1 \neq 0$ we use identities (4.9)–(4.10) with the same above reasoning. The proof is now complete. ∎

COROLLARY 4.1. *Let us suppose that* $u_0 \in D(A^{\rho})$ *and* $u_1 \in D(A^{\rho-1})$. *Then we have that the solution* u *of Eq.* (4.1) *satisfies*

$$u(t) \in D(A^{\rho+1}), \qquad u_t(t) \in D(A^{\rho})\ \forall t > 0.$$

Proof. With the same notation as in Theorem 4.1, we have to prove that $u_1(t) \in D(A^{\rho+1})$, $\forall t \geq \delta > 0$. From Lemma 4.1 and Lemma 4.2, it follows that for any $c > 0$ there exists $j_1 \in \mathbb{N}$ such that

$$\sum_{j=j_1}^{\infty} \lambda_j^{2(1+\rho)} \left| h_j(0) d_{1j} e^{r_{1j}t} \right|^2 < c^2 e^{2c\delta} \sum_{j=j_1}^{\infty} \lambda_j^{2\rho} h_j(0)^2 < +\infty,$$

where $\lambda_j d_{1j} < c$ and $r_{1j} < c$. From this our conclusion follows. ∎

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